

# Kadomtsev-Petviashvili equation in Relativistic Fluid Dynamics

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## Abstract

The Kadomtsev-Petviashvili (KP) nonlinear wave equation is the three dimensional generalization of the Korteweg-de Vries (KdV) equation. We show how to obtain the cylindrical KP (cKP) and cartesian KP in relativistic fluid dynamics. The obtained KP equations describe the evolution of perturbations in the baryon density in a strongly interacting quark gluon plasma (sQGP) at zero temperature. We also show the analytical solitary wave solution for the KP equations in both cases.

## I. INTRODUCTION

The Kadomtsev-Petviashvili equation (KP equation) [1] is a nonlinear wave equation in three spatial and one temporal coordinate. It is the generalization of the Korteweg-de Vries (KdV) equation to higher dimensions. The KP describes the evolution of long waves of small amplitudes with weak dependence on the transverse coordinates.

The KP equation has been found with the application of the reductive perturbation method [2] to several different problems such as the propagation of solitons in multicomponent plasmas and dust acoustic waves in hot dust plasmas [3–14].

The main goal of this work is to apply the reductive perturbation method [2–14] to relativistic fluid dynamics [15, 16] in cylindrical and cartesian coordinates to obtain the KP equation. We find that the transverse perturbations in relativistic fluid dynamics may generate three dimensional solitary waves.

Relativistic hydrodynamics is often applied to the study of quark matter in astrophysics and heavy ion collisions. Therefore we shall consider an equation of state (EOS) derived from QCD [17]. The obtained energy density and pressure contain derivative terms and a wave equation with a dispersive term such as KdV or KP emerges from the formalism. In [18], we have performed a similar study in one dimension and found a KdV equation. The present work is an extension of [18] to three dimensions.

Previous studies on one-dimensional nonlinear waves in cold and warm nuclear matter can be found in [19–25].

This text is organized as follows. In the next section we review the basic formulas of relativistic hydrodynamics. In section III we derive the KP equation in detail. In section IV we solve these KP equations analytically and in Section V we present some conclusions.

## II. RELATIVISTIC FLUID DYNAMICS

For a detailed study in relativistic hydrodynamics we suggest the references [15, 16].

The relativistic version of the Euler equation [15, 16, 18, 23] is given by:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{(\varepsilon + p)\gamma^2} \left( \vec{\nabla} p + \vec{v} \frac{\partial p}{\partial t} \right) \quad (1)$$

and the relativistic version of the continuity equation for the baryon density is [15]:

$$\partial_\nu j_B^\nu = 0 \quad (2)$$

Since  $j_B^\nu = u^\nu \rho_B$  the above equation can be rewritten as [18, 23]:

$$\frac{\partial \rho_B}{\partial t} + \gamma^2 v \rho_B \left( \frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla} v \right) + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0 \quad (3)$$

where  $\gamma = (1 - v^2)^{-1/2}$  is the Lorentz factor. In this work we employ the natural units  $\hbar = 1$ ,  $c = 1$ .

Recently [17] we have obtained an EOS for the strongly interacting quark gluon plasma (sQGP) at zero temperature. We performed a gluon field separation in “soft” and “hard” components, which correspond to low and high momentum components, respectively. In this approach the soft gluon fields are replaced by the in-medium gluon condensates. The hard gluon fields are treated in a mean field approximation and contribute with derivative terms in the equations of motion. Such equations solved properly may provide the time and space dependence of the quark (or baryon) density [17, 26]. We consider a system with massless quarks. The energy density is then given by [17, 26]:

$$\begin{aligned} \varepsilon = & \left( \frac{27g^2}{16m_G^2} \right) \rho_B^2 + \left( \frac{27g^2}{16m_G^4} \right) \rho_B \vec{\nabla}^2 \rho_B + \left( \frac{27g^2}{16m_G^6} \right) \rho_B \vec{\nabla}^2 (\vec{\nabla}^2 \rho_B) \\ & + \left( \frac{27g^2}{16m_G^8} \right) \vec{\nabla}^2 \rho_B \vec{\nabla}^2 (\vec{\nabla}^2 \rho_B) + \mathcal{B}_{QCD} + 3 \frac{\gamma_Q}{2\pi^2} \frac{k_F^4}{4} \end{aligned} \quad (4)$$

and the pressure is:

$$\begin{aligned} p = & \left( \frac{27g^2}{16m_G^2} \right) \rho_B^2 + \left( \frac{9g^2}{4m_G^4} \right) \rho_B \vec{\nabla}^2 \rho_B - \left( \frac{9g^2}{8m_G^6} \right) \rho_B \vec{\nabla}^2 (\vec{\nabla}^2 \rho_B) \\ & - \left( \frac{9g^2}{16m_G^4} \right) \vec{\nabla} \rho_B \cdot \vec{\nabla} \rho_B + \left( \frac{9g^2}{16m_G^6} \right) \vec{\nabla}^2 \rho_B \vec{\nabla}^2 \rho_B - \left( \frac{9g^2}{8m_G^8} \right) \vec{\nabla}^2 \rho_B \vec{\nabla}^2 (\vec{\nabla}^2 \rho_B) \\ & - \left( \frac{9g^2}{16m_G^8} \right) \vec{\nabla} (\vec{\nabla}^2 \rho_B) \cdot \vec{\nabla} (\vec{\nabla}^2 \rho_B) - \left( \frac{9g^2}{8m_G^6} \right) \vec{\nabla} \rho_B \cdot \vec{\nabla} (\vec{\nabla}^2 \rho_B) \\ & - \mathcal{B}_{QCD} + \frac{\gamma_Q}{2\pi^2} \frac{k_F^4}{4} \end{aligned} \quad (5)$$

In (4) and (5)  $\gamma_Q$  is the quark degeneracy factor  $\gamma_Q = 2(\text{spin}) \times 3(\text{flavor}) = 6$  where  $k_F$  is the Fermi momentum defined by the baryon number density:

$$\rho_B = \frac{\gamma_Q}{6\pi^2} k_F^3 \quad (6)$$

The other parameters  $g$ ,  $m_G$  and  $\mathcal{B}_{QCD}$  are the coupling of the hard gluons, the dynamical gluon mass and the bag constant in terms of the gluon condensate, respectively.

### III. THE KP EQUATION

We now combine the equations (1) and (3) to obtain the KP equation which governs the space-time evolution of the perturbation in the baryon density using the EOS given by (4) and (5). As mentioned above, we use the reductive perturbation method [2–9, 11–14]. Essentially, this formalism consists in expanding both (1) and (3) in powers of a small parameter  $\sigma$ . In the following subsections we present the formalism.

We start with the cylindrical KP (cKP) because it can be converted into an ordinary KdV equation (see section V) and similar transverse perturbations considering the two-dimensional cylindrical case have also been considered in our previous work [27].

#### A. Three-dimensional cylindrical coordinates

The field velocity of the relativistic fluid is:

$$\vec{v} = \vec{v}(r, \varphi, z, t) = \vec{v}_r(r, \varphi, z, t) + \vec{v}_\varphi(r, \varphi, z, t) + \vec{v}_z(r, \varphi, z, t)$$

and so  $|\vec{v}| = \sqrt{v_r^2 + v_\varphi^2 + v_z^2}$ .

Rewrite the equations (1) and (3) in dimensionless variables. The perturbations in baryon density occur upon a background of density  $\rho_0$  (the reference baryon density). It is convenient to write the baryon density as the dimensionless quantity:

$$\hat{\rho}(r, \varphi, z, t) = \frac{\rho_B(r, \varphi, z, t)}{\rho_0} \quad (7)$$

and similarly the velocity field as:

$$\hat{v} = \frac{v}{c_s} \quad (8)$$

where  $c_s$  is the speed of sound. The components of the velocity are:

$$\hat{v}_r(r, \varphi, z, t) = \frac{v_r(r, \varphi, z, t)}{c_s} \quad , \quad \hat{v}_\varphi(r, \varphi, z, t) = \frac{v_\varphi(r, \varphi, z, t)}{c_s}$$

and

$$\hat{v}_z(r, \varphi, z, t) = \frac{v_z(r, \varphi, z, t)}{c_s} \quad (9)$$

We next change variables from the space  $(r, \varphi, z, t)$  to the space  $(R, \Phi, Z, T)$  using the “stretched coordinates”:

$$R = \frac{\sigma^{1/2}}{L}(r - c_s t) \quad , \quad \Phi = \sigma^{-1/2}\varphi \quad , \quad Z = \frac{\sigma}{L}z \quad , \quad T = \frac{\sigma^{3/2}}{L}c_s t \quad (10)$$

where  $L$  is a typical scale of the problem.

The next step is the expansion of the dimensionless variables in powers of the small parameter  $\sigma$ :

$$\hat{\rho} = 1 + \sigma\rho_1 + \sigma^2\rho_2 + \sigma^3\rho_3 + \dots \quad (11)$$

$$\hat{v}_r = \sigma v_{r1} + \sigma^2 v_{r2} + \sigma^3 v_{r3} + \dots \quad (12)$$

$$\hat{v}_\varphi = \sigma^{3/2} v_{\varphi1} + \sigma^{5/2} v_{\varphi2} + \sigma^{7/2} v_{\varphi3} + \dots \quad (13)$$

$$\hat{v}_z = \sigma^{3/2} v_{z1} + \sigma^{5/2} v_{z2} + \sigma^{7/2} v_{z3} + \dots \quad (14)$$

$$\hat{\rho}^{4/3} = [1 + (\sigma\rho_1 + \sigma^2\rho_2 + \dots)]^{4/3} \cong 1 + \frac{4}{3}\sigma\rho_1 + \frac{4}{3}\sigma^2\rho_2 + \dots \quad (15)$$

$$\hat{\rho}^{1/3} = [1 + (\sigma\rho_1 + \sigma^2\rho_2 + \dots)]^{1/3} \cong 1 + \frac{1}{3}\sigma\rho_1 + \frac{1}{3}\sigma^2\rho_2 + \dots \quad (16)$$

Finally we neglect terms proportional to  $\sigma^n$  for  $n > 2$  and organize the equations as series in powers of  $\sigma$ ,  $\sigma^{3/2}$  and  $\sigma^2$ .

From the Euler equation (1) we find for the radial component:

$$\begin{aligned} & \sigma \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{r1}}{\partial R} + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_1}{\partial R} \right\} \\ & + \sigma^2 \left\{ \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_2}{\partial R} - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{r2}}{\partial R} \right. \\ & \quad + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \left( \frac{\partial v_{r1}}{\partial T} + v_{r1} \frac{\partial v_{r1}}{\partial R} \right) \\ & \quad + \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \rho_1 \frac{\partial \rho_1}{\partial R} + \pi^{2/3} \rho_0^{4/3} \frac{\rho_1}{3} \frac{\partial \rho_1}{\partial R} \\ & \quad - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) 2c_s^2 + 4\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \rho_1 \frac{\partial v_{r1}}{\partial R} \\ & \quad \left. - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + \pi^{2/3} \rho_0^{4/3} c_s^2 \right] v_{r1} \frac{\partial \rho_1}{\partial R} + \left( \frac{9g^2 \rho_0^2}{4m_G^4 L^2} \right) \frac{\partial^3 \rho_1}{\partial R^3} \right\} = 0 \end{aligned} \quad (17)$$

For the angular component:

$$\begin{aligned} & \sigma^{3/2} \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{\varphi1}}{\partial R} \right. \\ & \quad \left. + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{1}{T} \frac{\partial \rho_1}{\partial \Phi} \right\} = 0 \end{aligned} \quad (18)$$

and for the component in the  $z$  direction:

$$\sigma^{3/2} \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{z1}}{\partial R} + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_1}{\partial Z} \right\} = 0 \quad (19)$$

Performing the same calculations for the continuity equation (3) we find:

$$\begin{aligned} & \sigma \left\{ \frac{\partial v_{r_1}}{\partial R} - \frac{\partial \rho_1}{\partial R} \right\} + \sigma^2 \left\{ \frac{\partial v_{r_2}}{\partial R} - \frac{\partial \rho_2}{\partial R} + \frac{\partial \rho_1}{\partial T} \right. \\ & \left. + \rho_1 \frac{\partial v_{r_1}}{\partial R} + v_{r_1} \frac{\partial \rho_1}{\partial R} - c_s^2 v_{r_1} \frac{\partial v_{r_1}}{\partial R} + \frac{v_{r_1}}{T} + \frac{\partial v_{z_1}}{\partial Z} + \frac{1}{T} \frac{\partial v_{\varphi_1}}{\partial \Phi} \right\} = 0 \end{aligned} \quad (20)$$

In the last four equations each bracket must vanish independently and so  $\{\dots\} = 0$ . From the terms proportional to  $\sigma$  we obtain the identity:

$$\left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 = \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} = A \quad (21)$$

which defines the constant  $A$  and from which we obtain the speed of sound for a given background density  $\rho_0$ :

$$c_s^2 = \frac{\left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3}}{\left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + 3\pi^{2/3} \rho_0^{4/3}} \quad (22)$$

and also

$$\rho_1 = v_{r_1} \quad (23)$$

From the terms proportional to  $\sigma^{3/2}$  using the  $A$  constant we find:

$$\frac{\partial v_{\varphi_1}}{\partial R} = \frac{1}{T} \frac{\partial \rho_1}{\partial \Phi} \quad (24)$$

and

$$\frac{\partial v_{z_1}}{\partial R} = \frac{\partial \rho_1}{\partial Z} \quad (25)$$

Inserting the results (21), (23), (24) and (25) into the terms proportional to  $\sigma^2$  in (17) and (20), we find after some algebra, the cylindrical Kadomtsev-Petviashvili (cKP) equation [11]:

$$\begin{aligned} & \frac{\partial}{\partial R} \left\{ \frac{\partial \rho_1}{\partial T} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] \rho_1 \frac{\partial \rho_1}{\partial R} \right. \\ & \left. + \left[ \frac{9g^2 \rho_0^2}{8m_G^4 L^2 A} \right] \frac{\partial^3 \rho_1}{\partial R^3} + \frac{\rho_1}{2T} \right\} + \frac{1}{2T^2} \frac{\partial^2 \rho_1}{\partial \Phi^2} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Z^2} = 0 \end{aligned} \quad (26)$$

From the second identity of (21) we may write

$$\left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) = A - \pi^{2/3} \rho_0^{4/3} \quad (27)$$

Inserting (27) in the coefficient of the nonlinear term in (26) the cKP becomes:

$$\begin{aligned} \frac{\partial}{\partial R} \left\{ \frac{\partial \rho_1}{\partial T} + \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] \rho_1 \frac{\partial \rho_1}{\partial R} + \left[ \frac{9g^2 \rho_0^2}{8m_G^4 L^2 A} \right] \frac{\partial^3 \rho_1}{\partial R^3} + \frac{\rho_1}{2T} \right\} \\ + \frac{1}{2T^2} \frac{\partial^2 \rho_1}{\partial \Phi^2} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Z^2} = 0 \end{aligned} \quad (28)$$

Returning this cKP equation to the three dimension cylindrical space yields:

$$\begin{aligned} \frac{\partial}{\partial r} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{9g^2 \rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} + \frac{\hat{\rho}_1}{2t} \right\} \\ + \frac{1}{2c_s t^2} \frac{\partial^2 \hat{\rho}_1}{\partial \varphi^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \end{aligned} \quad (29)$$

which is the cKP equation for the second term of the expansion (11), the small perturbation given by  $\hat{\rho}_1 \equiv \sigma \rho_1$ .

## B. Three-dimensional cartesian coordinates

We now write the field velocity of the relativistic fluid as:

$$\vec{v} = \vec{v}(x, y, t) = \vec{v}_x(x, y, t) + \vec{v}_y(x, y, t) + \vec{v}_z(x, y, t)$$

and so  $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ .

We follow the same steps described in the cylindrical case to obtain the KP equation:

1) Rewrite the equations (1) and (3) in dimensionless variables:

$$\hat{\rho}(x, y, z, t) = \frac{\rho_B(x, y, z, t)}{\rho_0} \quad (30)$$

$$\hat{v} = \frac{v}{c_s} \quad (31)$$

and so:

$$\hat{v}_x(x, y, z, t) = \frac{v_x(x, y, z, t)}{c_s}, \quad \hat{v}_y(x, y, z, t) = \frac{v_y(x, y, z, t)}{c_s}$$

and

$$\hat{v}_z(r, \varphi, z, t) = \frac{v_z(r, \varphi, z, t)}{c_s} \quad (32)$$

2) Take the equations (1) and (3) (now in dimensionless variables) from the space  $(x, y, z, t)$  to the space  $(X, Y, Z, T)$  using the “stretched coordinates”:

$$X = \frac{\sigma^{1/2}}{L}(x - c_s t), \quad Y = \frac{\sigma}{L}y, \quad Z = \frac{\sigma}{L}z, \quad T = \frac{\sigma^{3/2}}{L}c_s t \quad (33)$$

3) Perform the expansions of the dimensionless variables:

$$\hat{\rho} = 1 + \sigma \rho_1 + \sigma^2 \rho_2 + \sigma^3 \rho_3 + \dots \quad (34)$$

$$\hat{v}_x = \sigma v_{x_1} + \sigma^2 v_{x_2} + \sigma^3 v_{x_3} + \dots \quad (35)$$

$$\hat{v}_y = \sigma^{3/2} v_{y_1} + \sigma^2 v_{y_2} + \sigma^{5/2} v_{y_3} + \dots \quad (36)$$

$$\hat{v}_z = \sigma^{3/2} v_{z_1} + \sigma^2 v_{z_2} + \sigma^{5/2} v_{z_3} + \dots \quad (37)$$

$$\hat{\rho}^{4/3} \cong 1 + \frac{4}{3} \sigma \rho_1 + \frac{4}{3} \sigma^2 \rho_2 + \dots \quad (38)$$

$$\hat{\rho}^{1/3} \cong 1 + \frac{1}{3} \sigma \rho_1 + \frac{1}{3} \sigma^2 \rho_2 + \dots \quad (39)$$

4) Neglect terms proportional to  $\sigma^n$  for  $n > 2$  and organize the equations as series in powers of  $\sigma$ ,  $\sigma^{3/2}$  and  $\sigma^2$ .

After these manipulations the  $x$ ,  $y$  and  $z$  components of Euler equation are:

$$\begin{aligned} & \sigma \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{x_1}}{\partial X} + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_1}{\partial X} \right\} \\ & + \sigma^2 \left\{ \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_2}{\partial X} - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{x_2}}{\partial X} \right. \\ & \quad + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \left( \frac{\partial v_{x_1}}{\partial T} + v_{x_1} \frac{\partial v_{x_1}}{\partial X} \right) \\ & \quad + \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \rho_1 \frac{\partial \rho_1}{\partial X} + \pi^{2/3} \rho_0^{4/3} \frac{\rho_1}{3} \frac{\partial \rho_1}{\partial X} \\ & \quad - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) 2c_s^2 + 4\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \rho_1 \frac{\partial v_{x_1}}{\partial X} \\ & \quad \left. - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + \pi^{2/3} \rho_0^{4/3} c_s^2 \right] v_{x_1} \frac{\partial \rho_1}{\partial X} + \left( \frac{9g^2 \rho_0^2}{4m_G^4 L^2} \right) \frac{\partial^3 \rho_1}{\partial X^3} \right\} = 0 \end{aligned} \quad (40)$$

$$\begin{aligned} & \sigma^{3/2} \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{y_1}}{\partial X} + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_1}{\partial Y} \right\} \\ & + \sigma^2 \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{y_2}}{\partial X} \right\} = 0 \end{aligned} \quad (41)$$

and

$$\begin{aligned} & \sigma^{3/2} \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{z_1}}{\partial X} + \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) + \pi^{2/3} \rho_0^{4/3} \right] \frac{\partial \rho_1}{\partial Z} \right\} \\ & + \sigma^2 \left\{ - \left[ \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) c_s^2 + 3\pi^{2/3} \rho_0^{4/3} c_s^2 \right] \frac{\partial v_{z_2}}{\partial X} \right\} = 0 \end{aligned} \quad (42)$$



For the continuity equation we obtain:

$$\sigma \left\{ \frac{\partial v_{x_1}}{\partial X} - \frac{\partial \rho_1}{\partial X} \right\} + \sigma^2 \left\{ \frac{\partial v_{x_2}}{\partial X} - \frac{\partial \rho_2}{\partial X} + \frac{\partial \rho_1}{\partial T} + \rho_1 \frac{\partial v_{x_1}}{\partial X} + v_{x_1} \frac{\partial \rho_1}{\partial X} - c_s^2 v_{x_1} \frac{\partial v_{x_1}}{\partial X} + \frac{\partial v_{y_1}}{\partial Y} + \frac{\partial v_{z_1}}{\partial Y} \right\} = 0 \quad (43)$$

Again, in the last four equations each bracket must vanish independently. From the terms proportional to  $\sigma$  we obtain the same  $A$  constant as in the cylindrical case given by (21), the same expression for the speed of sound (22) and

$$\rho_1 = v_{x_1} \quad (44)$$

From the terms proportional to  $\sigma^{3/2}$  we find

$$\frac{\partial v_{y_1}}{\partial X} = \frac{\partial \rho_1}{\partial Y} \quad (45)$$

and

$$\frac{\partial v_{z_1}}{\partial X} = \frac{\partial \rho_1}{\partial Z} \quad (46)$$

In (41) and (42) we have from the terms proportional to  $\sigma^2$ :

$$\frac{\partial v_{y_2}}{\partial X} = \frac{\partial v_{z_2}}{\partial X} = 0 \quad (47)$$

Inserting the results (21), (44), (45), (46) and (47) into the terms proportional to  $\sigma^2$  in (40) and (43), we find after some algebra, the Kadomtsev-Petviashvili (KP) equation [6, 14]:

$$\begin{aligned} \frac{\partial}{\partial X} \left\{ \frac{\partial \rho_1}{\partial T} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] \rho_1 \frac{\partial \rho_1}{\partial X} \right. \\ \left. + \left[ \frac{9g^2 \rho_0^2}{8m_G^4 L^2 A} \right] \frac{\partial^3 \rho_1}{\partial X^3} \right\} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Y^2} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Z^2} = 0 \end{aligned} \quad (48)$$

Inserting (27) in (48), the KP with simplified coefficient for the nonlinear term is given by:

$$\frac{\partial}{\partial X} \left\{ \frac{\partial \rho_1}{\partial T} + \left[ \frac{3}{2} (1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] \rho_1 \frac{\partial \rho_1}{\partial X} + \left[ \frac{9g^2 \rho_0^2}{8m_G^4 L^2 A} \right] \frac{\partial^3 \rho_1}{\partial X^3} \right\} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Y^2} + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial Z^2} = 0 \quad (49)$$

Rewriting this KP equation back in the three dimensional cartesian space we find:

$$\frac{\partial}{\partial x} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{3}{2} (1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{9g^2 \rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} \right\} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial y^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \quad (50)$$

which is the KP equation for the small perturbation  $\hat{\rho}_1 \equiv \sigma \rho_1$ , the second term of the expansion (34).

### C. Some particular cases

In the one dimensional cartesian relativistic fluid dynamics we have  $\vec{v} = \vec{v}(x, t)$  and  $\rho_B = \rho_B(x, t)$ . Repeating all the steps of the last subsection for one dimension, the reductive perturbation method reduces to the formalism previously used in [19–25] and we find the following particular cases of (50):

(I) Neglecting the  $y$  and  $z$  dependence, the (50) becomes the Korteweg-de Vries equation (KdV) similar to the KdV found in [18]:

$$\begin{aligned} \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} \\ + \left[ \frac{9g^2 \rho_0^2 c_s}{8m_G^4 A} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} \Big\} = 0 \end{aligned} \quad (51)$$

Taking the limit  $m_G \rightarrow \infty$  we obtain from (21) and (22):

$$A = \pi^{2/3} \rho_0^{4/3} \quad , \quad c_s^2 = \frac{1}{3}$$

and (51) becomes:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \frac{2}{3} c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0 \quad (52)$$

and we recover exactly the result found in [23], the so called breaking wave equation for  $\hat{\rho}_1$  at zero temperature in the QGP with the MIT equation of state.

(II) Neglecting the spatial derivatives in (4) and (5), we obtain (51) reduces to:

$$\frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{(2 - c_s^2)}{2} - \left( \frac{27g^2 \rho_0^2}{8m_G^2} \right) \frac{(2c_s^2 - 1)}{2A} - \frac{\pi^{2/3} \rho_0^{4/3}}{A} \left( c_s^2 - \frac{1}{6} \right) \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} = 0 \quad (53)$$

which is also a breaking wave equation for  $\hat{\rho}_1$  with the  $\rho_0$ ,  $m_G$  and  $g$  dependence in its coefficients.

### IV. NON-RELATIVISTIC LIMIT

The non-relativistic version of the continuity equation (1) is given by [15, 16]:

$$\frac{\partial \rho_B}{\partial t} + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0 \quad (54)$$

and for the Euler equation we have (3) [15, 16]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \left( \frac{1}{\rho} \right) \vec{\nabla} p \quad (55)$$

where  $\rho$  is the volumetric density of fluid matter. In this work we study perturbations for baryon density in the sQGP fluid, so we define the “effective baryon mass in sQGP”:

$$\rho = \mathcal{M}\rho_B \quad (56)$$

which will be determined latter. Substituting (56) in (55) we find the non-relativistic version for the Euler equation in sQGP:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\left(\frac{1}{\mathcal{M}\rho_B}\right)\vec{\nabla}p \quad (57)$$

Performing all the calculations described in the last section for the combination of (54) and (57) we find the cKP equation in non-relativistic hydrodynamics:

$$\begin{aligned} \frac{\partial}{\partial r} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{3}{2} - \frac{\pi^{2/3} \rho_0^{1/3}}{3\mathcal{M}c_s^2} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial r} + \left[ \frac{9g^2 \rho_0}{8\mathcal{M}m_G^4 c_s} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} + \frac{\hat{\rho}_1}{2t} \right\} \\ + \frac{1}{2c_s t^2} \frac{\partial^2 \hat{\rho}_1}{\partial \varphi^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \end{aligned} \quad (58)$$

and the KP equation in three-dimensional cartesian coordinates:

$$\frac{\partial}{\partial x} \left\{ \frac{\partial \hat{\rho}_1}{\partial t} + c_s \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{3}{2} - \frac{\pi^{2/3} \rho_0^{1/3}}{3\mathcal{M}c_s^2} \right] c_s \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial x} + \left[ \frac{9g^2 \rho_0}{8\mathcal{M}m_G^4 c_s} \right] \frac{\partial^3 \hat{\rho}_1}{\partial x^3} \right\} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial y^2} + \frac{c_s}{2} \frac{\partial^2 \hat{\rho}_1}{\partial z^2} = 0 \quad (59)$$

which are the non-relativistic versions of (29) and (50) respectively. During the derivation in both cases we find from the terms proportional to  $\sigma$  in the Euler equation:

$$\mathcal{M} = \left( \frac{27g^2 \rho_0}{8m_G^2 c_s^2} \right) + \frac{\pi^{2/3} \rho_0^{1/3}}{c_s^2} \quad (60)$$

We end this section mentioning that it is possible to obtain (58) and (59) directly from (29) and (50) respectively, performing the two non-relativistic approximations:

$$a) \quad c_s^2 \rightarrow 0 \quad (61)$$

and

$$b) \quad A = \mathcal{M}\rho_0 c_s^2 \quad (62)$$

where  $A$  is given by (21) and  $\mathcal{M}$  by (60).

## V. THE ANALYTICAL SOLITON-LIKE SOLUTION FOR CKP AND KP EQUATION

In this section we present the analytical soliton-like solution to the cKP and KP equation given by (29) and (50) respectively. The KP equation is an integrable system in three dimensions in the same way as the KdV is in one dimension. We introduce a set of coordinates that transforms (29) in an ordinary KdV, which is a solvable equation, and we also present the analytical solution of (50). In order to simplify the notation in equations (29) and (50) we define the constants:

$$\alpha \equiv \left[ \frac{3}{2}(1 - c_s^2) - \frac{\pi^{2/3} \rho_0^{4/3}}{3A} \right] c_s \quad (63)$$

and

$$\beta \equiv \left[ \frac{9g^2 \rho_0^2 c_s}{8m_G^4 A} \right] \quad (64)$$

Considering the following coordinates [5, 9, 11, 28]:

$$\xi = r - \frac{c_s \varphi^2 t}{2} + z \quad \text{and} \quad \tau = t \quad (65)$$

we have:

$$\begin{aligned} \frac{\partial}{\partial r} &\rightarrow \frac{\partial}{\partial \xi} \quad , \quad \frac{\partial^3}{\partial r^3} \rightarrow \frac{\partial^3}{\partial \xi^3} \quad , \quad \frac{\partial^2}{\partial z^2} \rightarrow \frac{\partial^2}{\partial \xi^2} \quad , \\ \frac{\partial^2}{\partial \varphi^2} &\rightarrow c_s^2 \varphi^2 t^2 \frac{\partial^2}{\partial \xi^2} - c_s t \frac{\partial}{\partial \xi} \quad , \quad \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} - \frac{c_s \varphi^2}{2} \frac{\partial}{\partial \xi} \end{aligned} \quad (66)$$

and

$$\hat{\rho}_1(r, \varphi, z, t) \rightarrow \hat{\rho}_1(\xi, \tau) \quad (67)$$

Using (66) and (67) in (29) we find the KdV equation in the  $(\xi, \tau)$  space:

$$\frac{\partial \hat{\rho}_1}{\partial \tau} + \left( \frac{3c_s}{2} \right) \frac{\partial \hat{\rho}_1}{\partial \xi} + \alpha \hat{\rho}_1 \frac{\partial \hat{\rho}_1}{\partial \xi} + \beta \frac{\partial^3 \hat{\rho}_1}{\partial \xi^3} = 0 \quad (68)$$

which has the analytical soliton solution given by:

$$\hat{\rho}_1(\xi, \tau) = \frac{3(u - 3c_s/2)}{\alpha} \text{sech}^2 \left[ \sqrt{\frac{(u - 3c_s/2)}{4\beta}} \left( \xi - u\tau \right) \right] \quad (69)$$

The exact analytical soliton solution of (29) in three cylindrical coordinates is obtained substituting (65) in (69):

$$\hat{\rho}_1(r, \varphi, z, t) = \frac{3(u - 3c_s/2)}{\alpha} \text{sech}^2 \left\{ \sqrt{\frac{(u - 3c_s/2)}{4\beta}} \left[ r + z - \left( u + \frac{c_s \varphi^2}{2} \right) t \right] \right\} \quad (70)$$

where  $u$  is a supersonic parameter which satisfies  $u > \frac{3c_s}{2}$  and the phase velocity given by  $u + \frac{c_s \varphi^2}{2}$  is angle dependent.

The exact analytical soliton solution of the KP equation (50) is given by [6, 29, 30]:

$$\hat{\rho}_1(x, y, z, t) = \frac{3(U - 2c_s)}{\alpha} \text{sech}^2 \left[ \sqrt{\frac{(U - 2c_s)}{4\beta}} (x + y + z - Ut) \right] \quad (71)$$

where again we have a supersonic parameter  $U$  such that  $U > 2c_s$ .

## VI. CONCLUSIONS

We have described in detail how to obtain a KP equation in three dimensions in cylindrical and cartesian coordinates in the context of relativistic fluid dynamics of a cold quark gluon plasma. To this end, we have used the equation of state derived from QCD in [17]. The resulting nonlinear relativistic wave equations are for small perturbation in the baryon density.

For the cartesian KP the exact soliton solution is a supersonic bump keeping its shape without deformation. The cartesian KP contains some particular cases such as KdV and the breaking wave equation already encountered in our previous works [18, 23]. For the cylindrical KP (cKP) we also have an exact supersonic soliton solution which deforms slightly as time goes on due to the angular dependence in the phase.

We conclude that relativistic fluid dynamics supports nonlinear solitary waves even with the inclusion of transverse perturbations in cylindrical and cartesian geometry.

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